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Field of a test charge in an anisotropic plasma

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The field of a test charge has been determined correct to second order in the anisotropy parameter and comparison has been made with previous results.

1. INTRODUCTION

The screening of the Coulomb field of a test charge inside a plasma is well known. The usual $\exp(-r/\lambda_D)$ screening occurs only when the test charge is at rest and the plasma is isotropic (Thomson 1962). When either of these two conditions is violated the field may behave as r^{-n} for large r (Joyce & Montgomery 1967; Montgomery, Joyce & Sugihara 1968). It has been shown by Montgomery *et al* that in a plasma with unequal longitudinal and transverse velocities $V_{\|}$ and $V_{\perp}$, the test charge potential varies as r^{-3} irrespective of the value of the anisotropy parameter $\alpha = V_{\|}^2/V_{\perp}^2$. Thus one fails to recover from their results the usual screened Coulomb potential for $\alpha = 0$ and consequently doubts arise regarding the validity of their results for small value of α. In this paper we have presented a method of calculation in which the above mentioned difficulty is circumvented and it becomes possible to determine the potential exactly to any desired order in α.

8. CALCULATION AND RESULTS

The field of a stationary test charge q situated at the origin of the co-ordinate system is given by

$$\phi(\boldsymbol{r}) = \lim_{\varepsilon \to 0+} \int d\boldsymbol{k} \; \frac{qe^{-\varepsilon|k|}}{2\pi^2} \cdot \frac{e^{i\boldsymbol{k}.\boldsymbol{r}}}{D(\boldsymbol{k})}, \qquad \ldots \quad (1)$$

when $D(\boldsymbol{k})$ is the well known plasma dielectric function with $\omega = 0$. For a uniform neutral plasma with ions at rest and the electron distribution function given by (Montgomery *et al* 1968).

$$f_e(\boldsymbol{v}) = \frac{\exp\left[-\dfrac{v^2}{2v^2_{\perp}} - (\boldsymbol{v}.\boldsymbol{a})^2\left(\dfrac{1}{2v_{\|}^2} - \dfrac{1}{2v_{\perp}^2}\right)\right]}{(2\pi)^{3/2} V_{\perp}^2 V_{\|}}, \qquad \ldots \quad (2)$$

where $\boldsymbol{a}$ is a unit vector in any arbitrary direction, $D(\boldsymbol{k})$ is given by

$$D(\boldsymbol{k}) = 1 + k^{-2}\psi, \qquad \ldots \quad (3)$$

with

$$\psi = K_1^2\,[1-\alpha(\boldsymbol{a}.\hat{\boldsymbol{k}})^2], \qquad \ldots \quad (3a)$$

$$\hat{\boldsymbol{k}} = \boldsymbol{k}/k;\; K_\perp^2 = w_{pc}^2/v_\perp^2;\; \alpha = 1-v_{\|}^2 v_\perp^2. \qquad \ldots \quad (3b)$$

We choose a Cartesian frame with z-axis along $\boldsymbol{r}$ and $\boldsymbol{a}$ lying in the $x-z$ plane, inclined at an angle γ to the z-axis. If $\theta = \cos^{-1}\mu$ and ϕ denote the polar and azimuthal angles of $\boldsymbol{k}$, then

$$\boldsymbol{a}\cdot\hat{\boldsymbol{k}} = \mu\cos\gamma+(1-\mu^2)^{\frac{1}{2}}\sin\gamma\cos\phi. \qquad \ldots \quad (4)$$

Our task will now be to perform the threefold integration in eq. (1) with $D(\boldsymbol{k})$ given by the set of eqs. (3) and (4). The method employed by Montgomery *et al* (1968) to this end is effectively based on an expansion of $D(\boldsymbol{k})$ in a power series of $k^2[1-\alpha(\boldsymbol{a}.\hat{\boldsymbol{k}})^2]$. Evidently such an expansion cannot remain convergent in the entire domain of integration of k for any fixed $\boldsymbol{r}$, however large. Further the convergence of the series becomes definitely worse as $\alpha \to 0$. Consequently their results have the shortcoming stated earlier.

Transforming to spherical polar coordinates in k space eq. (1) may be written in the following form (Appendix A)

$$\phi(\boldsymbol{r}) = \lim_{\varepsilon\to 0+} \int d\boldsymbol{k}\, \frac{q\epsilon e^{-\varepsilon k}}{2\pi^2}\, \frac{e^{i\boldsymbol{k}.\boldsymbol{r}}}{k^2} - \frac{\epsilon K_\perp^2}{\pi^2} \int_{-\infty}^{+\infty} d\mathrm{k} \int_0^1 d\mu \int_0^{\pi/2} d\phi\, e^{ikr\mu}$$

$$\times\Bigg[\frac{1}{K_\perp^2+k^2\{1-\alpha(\mu\cos\gamma+(1-\mu^2)^{\frac{1}{2}}\sin\gamma\cos\phi)^2\}}$$

$$+\frac{1}{K_\perp^2+k^2\{1-\alpha(-\mu\cos\gamma+(1-\mu^2)^{\frac{1}{2}}\sin\gamma\cos\phi)^2\}}\Bigg]. \qquad \ldots \quad (5)$$

The first term is the usual Coulomb potential q/r. The second and the third integrals are evaluated by exapanding the denominator in a power series of α.

$$\Sigma\left(\frac{\alpha k^2}{k^2+K_\perp^2}\right)^n(\pm\mu\cos\gamma+(1-\mu^2)^{\frac{1}{2}}\sin\gamma\cos\phi)^{2n}.$$

which converges uniformly in the entire domain of integration over k for $0 \leqslant |\alpha| \leqslant 1$. This expansion has the further advantage that when we combine the corresponding terms from the second and the third integrals the terms containing the fractional powers of $(1-\mu^2)$ cancel out and each term in the resulting expression is integrable in closed form. Thus one can easily determine the test charge potential correct to any order in the anisotropy parameter α.

To first order in α, we have

$$\phi(\boldsymbol{r}) = \phi_0(\boldsymbol{r}) + \phi_1(\boldsymbol{r}).$$

$$= \frac{q}{r} - \frac{2qK_\perp^2}{\pi^2} \int_0^{\pi/2} d\phi \int_0^1 d\mu \int_{-\infty}^{+\infty} dk \, \frac{e^{ikr\mu}}{k^2+K_\perp^2}$$

$$\times \left[1 + \frac{\alpha k^2}{k^2+K_\perp^2} \, (\mu^2 \cos^2 \gamma + (1-\mu^2)^{\frac{1}{2}} \sin^2 \gamma \cos^2 \phi) \right]$$

$$= \frac{q}{r} e^{-k_\perp r} + \alpha q (1 - 3/2 \sin^2 \gamma) \left[\frac{2}{K_\perp^2 r^3} - \frac{1}{r} e^{-K_\perp r} \right.$$

$$\left. - \frac{3}{K_\perp r^2} e^{-K_\perp r} - \frac{1}{K_\perp^2 r^3} e^{-K_\perp r} \right] - 2\alpha q K_\perp^2 \cos^2 \gamma \, e^{-K_\perp r} \qquad \ldots \quad (6)$$

On making $\alpha \to 0$ we recover the screened term. The result of eq. (6) may be compared with the result ϕ_{MJS} (Montgomery *et al* 1968) which just equals the second term of eq. (6).

$$\phi_{MJS} = \frac{2\alpha q}{K_\perp^2 r^3} (1 - 3/2 \sin^2 \gamma). \qquad \ldots \quad (7)$$

One can see from eq. (6) how the contribution from the screened terms dwindles relative to the contribution from r^{-3} term as $\alpha \to 0$ or as $r \to \infty$. Finally we give the contributions from second order terms in α

$$\phi_2(\boldsymbol{r}) = - \frac{2qK_\perp^2}{\pi^2} \alpha^2 \int_0^{\pi/2} d\phi \int_0^1 d\mu \int_{-\infty}^{+\infty} dk \, \frac{k^4}{(k^2+K_\perp^2)^3} e^{ikr\mu}$$

$$\times [\mu^4 \cos^4 \gamma + 6\mu^2 (1-\mu^2) \cos^2 \gamma \sin^2 \gamma \cos^2 \phi + (1-\mu^2)^2 \sin^4 \gamma \cos^4 \phi].$$

$$= - \frac{q\alpha^2}{K_\perp^4 r^5} \left(1 - 5 \sin^2 \gamma + \frac{35}{8} \sin^4 \gamma \right) + q\alpha^2 K_\perp e^{-K_\perp r}$$

$$\left(1 - 5 \sin^2 \gamma + \frac{35}{8} \sin^4 \gamma \right) \left[\frac{48}{(K_\perp r)^5} + \frac{24}{(K_\perp r)^4} + \frac{24}{(K_\perp r)^3} + \frac{8}{(K_\perp r)^2} + \frac{2}{K_\perp r} \right]$$

$$+ \frac{q\alpha^2 K_\perp}{8} e^{-K_\perp r} (1 - 8 \sin^2 \gamma - \tfrac{1}{2} \sin^4 \gamma).$$

$$+ \frac{q\alpha^2 K_\perp^2 r}{8} e^{-K_\perp r} \left(1 - 2 \sin^2 \gamma + \frac{17}{2} \sin^4 \gamma \right). \qquad \ldots \quad (8)$$

Appendix A

$$\phi(\mathbf{r}) = \frac{q}{2\pi^2} \lim_{\varepsilon\to 0+} \int_0^{2\pi} d\phi \int_{-1}^{+1} d\mu \int_0^{\infty} dk\, e^{-\varepsilon k}\, e^{ikr\mu}$$

$$\left[1 - \frac{K_\perp^2}{K_\perp^2 + k^2(1-\alpha(\mathbf{a}.\mathbf{k})^2)}\right] \qquad \ldots \quad (A.1)$$

where $(\mathbf{a}.\mathbf{k})^2 = (\mu \cos\gamma + (1-\mu^2)^{\frac{1}{2}} \sin\gamma \cos\phi)^2$.

In the second term we split up the integrals over ϕ and μ and group them as shown below

$$\int_0^{2\pi} d\phi \int_{-1}^{+1} d\mu = \left[\int_0^{\pi} d\phi \int_0^1 d\mu + \int_{\pi}^{2\pi} d\phi \int_{-1}^{0} d\mu\right] + \left[\int_0^{\pi} d\phi \int_{-1}^{0} d\mu + \int_{\pi}^{2\pi} d\phi \int_0^1 d\mu\right]$$

$$\qquad\qquad (\mathrm{I}) \qquad\qquad (\mathrm{II}) \qquad\qquad (\mathrm{III}) \qquad\qquad (\mathrm{IV})$$

Combining (I) and (II) with suitable change of integration variables ($\phi \to \pi + \phi$, $\mu \to -\mu$ and $k \to -k$ in II) we get the second term in eq. (5) while (III) and (IV) together ($\mu \to -\mu$ and $k \to -k$ in III and $\phi \to \pi+\phi$ in IV) give the third term.

Appendix B

The integration over μ is done first followed by simple contour integration over k. We have

$$\int_{-\infty}^{+\infty} dk \frac{k^2}{(k^2+K_\perp^2)^2} \int_0^1 d\mu\, \mu^2 e^{ik\mu r}$$

$$= \frac{\pi}{K_\perp} e^{-K_\perp r}\left(2 + \frac{1}{K_\perp r} + \frac{3}{(K_\perp r)^2} + \frac{1}{(K_\perp r)^3}\right) - \frac{2\pi}{K_\perp^4 r^3} \qquad \ldots \quad (B.1)$$

$$\int_{-\infty}^{+\infty} dk \frac{k^2}{(k^2+K^2_\perp)^3} \int_0^1 d\mu (1-\mu^2) e^{ik\mu r}$$

$$= -\frac{\pi}{K_\perp} e^{-K_\perp r}\left(\frac{1}{K_\perp r} + \frac{3}{(K_\perp r)^2} + \frac{1}{(K_\perp r)^3}\right) + \frac{2\pi}{K_\perp^4 r^3} \qquad \ldots \quad (B.2)$$

$$\int_{-\infty}^{+\infty} dk \frac{k^4}{(k^2+K_\perp^2)^3} \int_0^1 d\mu\, \mu^4 e^{ik\mu r} = -\frac{\pi}{K_\perp} e^{-K_\perp r}\left(\frac{K_\perp r}{4} + 1\right.$$

$$\left. + \frac{2}{K_\perp r} + \frac{8}{(K_\perp r)^2} + \frac{24}{(K_\perp r)^3} + \frac{24}{(K_\perp r)^4} + \frac{48}{(K_\perp r)^5}\right) + \frac{24\pi}{K_\perp^6 r^5}. \qquad \ldots \quad (B.3)$$

$$\int_{-\infty}^{+\infty} dk \frac{k^4}{(k^2+K_\perp^2)^3} \int_0^1 d\mu\, \mu^2 e^{ik\mu r} = -\frac{\pi}{4} r e^{-K_\perp r} + \frac{\pi}{4K} e^{-K_\perp r} \quad \ldots \quad (B.4)$$

$$\int_{-\infty}^{+\infty} dk \frac{k^4}{(k^2+K_\perp^2)^3} \int_0^1 d\mu\, e^{ik\mu r} = -\frac{\pi}{4} r e^{-K_\perp r} + \frac{3\pi}{4K_\perp} e^{-K_\perp r} \quad \ldots \quad (B.5)$$

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